Matrix theory, Data science

and

Model reduction of large-scale systems

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PART II

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Outline

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2. The Loewner framework
   - Generalized inverses and rectangular systems
   - Illustrative examples
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The basics of Interpolatory Model Reduction

Summary. The study of interpolatory reduction methods can be subdivided in three areas:

(1) Interpolatory projections,
(2) the Loewner Framework, and
(3) $\mathcal{H}_2$ Optimal Reduction known as IRKA.

Below, we will discuss these three aspects, first as applied to linear systems. Subsequently, we will outline their generalization to bilinear and more generally non-linear systems.

Set-up. To start with, we consider linear, time-invariant systems

$$\mathbf{E}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \quad \mathbf{y} = \mathbf{C}\mathbf{x}, \tag{1}$$

assumed to be scalar for simplicity, i.e.

$$\mathbf{E},\ \mathbf{A} \in \mathbb{R}^{n\times n},\ \mathbf{B},\ \mathbf{C}^T \in \mathbb{R}^n.$$

We will denote this realization of the system by means of the quadruple $(\mathbf{C}, \mathbf{E}, \mathbf{A}, \mathbf{B})$. The associated transfer function is

$$\mathbf{H}(s) = \mathbf{C}\boldsymbol{\Phi}(s)\,\mathbf{B},$$

where $\boldsymbol{\Phi}$ denotes the resolvent

$$\boldsymbol{\Phi}(s) = (s\,\mathbf{E} - \mathbf{A})^{-1} \in \mathbb{C}^{n\times n}. \tag{2}$$

Interpolatory projectors (Skelton, Grimme-Van Dooren)

We introduce the following quantities which will play the role of projectors in the sequel. For this we need two sets of complex numbers namely μ_i, $i = 1, \cdots, q$, and λ_j, $j = 1, \cdots, k$, which we will refer to as *left* and *right* interpolation points:

$$\mathcal{R} = \begin{bmatrix} \mathbf{\Phi}(\lambda_1)\mathbf{B} & \cdots & \mathbf{\Phi}(\lambda_k)\mathbf{B} \end{bmatrix} \in \mathbb{C}^{n\times k} \quad \text{and} \quad \mathcal{O} = \begin{bmatrix} \mathbf{C\Phi}(\mu_1) \\ \vdots \\ \mathbf{C\Phi}(\mu_q) \end{bmatrix} \in \mathbb{C}^{q\times n}. \qquad (3)$$

These are called *the generalized controllability* and the *generalized observability* matrices.

Proposition.
With $\mathbf{\Lambda} = \text{diag}\,[\lambda_1, \ \cdots, \ \lambda_k]$, $\mathbf{M} = \text{diag}\,[\mu_1, \ \cdots, \ \mu_q]$, $\mathbf{e}_m = [1 \ \cdots \ 1]^T \in \mathbb{R}^m$,
the matrices $\mathcal{R}$ and $\mathcal{O}$ satisfy the Sylvester equations:

$$\boxed{\mathbf{E}\mathcal{R}\mathbf{\Lambda} - \mathbf{A}\mathcal{R} = \mathbf{B}\mathbf{e}_k^T} \quad \text{and} \quad \boxed{\mathbf{M}\mathcal{O}\mathbf{E} - \mathcal{O}\mathbf{A} = \mathbf{e}_q\mathbf{C}} \qquad (4)$$

Construction. For arbitrary k and q, the following relationships hold:

$$\hat{\mathbf{E}} = \mathcal{O}\mathbf{E}\mathcal{R} = -\begin{bmatrix} \frac{\mathbf{H}(\mu_1)-\mathbf{H}(\lambda_1)}{\mu_1-\lambda_1} & \cdots & \frac{\mathbf{H}(\mu_1)-\mathbf{H}(\lambda_k)}{\mu_1-\lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{H}(\mu_q)-\mathbf{H}(\lambda_1)}{\mu_q-\lambda_1} & \cdots & \frac{\mathbf{H}(\mu_q)-\mathbf{H}(\lambda_k)}{\mu_q-\lambda_k} \end{bmatrix} = -\mathbb{L} \in \mathbb{C}^{q\times k}, \quad (5)$$

$$\hat{\mathbf{A}} = \mathcal{O}\mathbf{A}\mathcal{R} = -\begin{bmatrix} \frac{\mu_1\mathbf{H}(\mu_1)-\lambda_1\mathbf{H}(\lambda_1)}{\mu_1-\lambda_1} & \cdots & \frac{\mu_1\mathbf{H}(\mu_1)-\lambda_k\mathbf{H}(\lambda_k)}{\mu_1-\lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mu_q\mathbf{H}(\mu_q)-\lambda_1\mathbf{H}(\lambda_1)}{\mu_q-\lambda_1} & \cdots & \frac{\mu_q\mathbf{H}(\mu_q)-\lambda_k\mathbf{H}(\lambda_k)}{\mu_q-\lambda_k} \end{bmatrix} = -\mathbb{L}_s \in \mathbb{C}^{q\times k}, \quad (6)$$

$$\hat{\mathbf{B}} = \mathcal{O}\mathbf{B} = \begin{bmatrix} \mathbf{H}(\mu_1) \\ \vdots \\ \mathbf{H}(\mu_q) \end{bmatrix} = \mathbf{V} \in \mathbb{C}^{q\times 1}, \quad \text{and} \quad (7)$$

$$\hat{\mathbf{C}} = \mathbf{C}\mathcal{R} = \begin{bmatrix} \mathbf{H}(\lambda_1) & \cdots & \mathbf{H}(\lambda_k) \end{bmatrix} = \mathbf{W} \in \mathbb{C}^{1\times k}. \quad (8)$$

The resulting quadruple $(\mathbf{W}, \mathbb{L}, \mathbb{L}_s, \mathbf{V})$ is called the *Loewner quadruple*.

Proposition. Upon multiplication of (4) with $\mathcal{O}$ on the left and $\mathcal{R}$ on the right, we obtain:

$$\boxed{\mathbb{L}_s - \mathbb{L}\,\mathbf{\Lambda} = \mathbf{V}\,\mathbf{R}} \quad \text{and} \quad \boxed{\mathbb{L}_s - \mathbf{M}\,\mathbb{L} = \mathbf{L}\,\mathbf{W}} \tag{9}$$

By adding/subtracting appropriate multiples of these expressions it follows that the Loewner quadruple satisfies the Sylvester equations

$$\mathbf{M}\mathbb{L} - \mathbb{L}\,\mathbf{\Lambda} = \mathbf{V}\mathbf{R} - \mathbf{L}\mathbf{W} \quad \text{and} \quad \mathbf{M}\,\mathbb{L}_s - \mathbb{L}_s\mathbf{\Lambda} = \mathbf{M}\mathbf{V}\mathbf{R} - \mathbf{L}\mathbf{W}\mathbf{\Lambda}.$$

Interpolation property of reduced systems.

Given the projectors $\mathbf{X}, \mathbf{Y} \in \mathbb{C}^{n\times k}$, let the reduced quantities be

$$\hat{\mathbb{L}} = \mathbf{X}^*\mathbb{L}\mathbf{Y},\ \hat{\mathbb{L}_s} = \mathbf{X}^*\mathbb{L}_s\mathbf{Y},\ \hat{\mathbf{V}} = \mathbf{X}^*\mathbf{V},\ \hat{\mathbf{L}} = \mathbf{X}^*\mathbf{L},\ \hat{\mathbf{W}} = \mathbf{W}\mathbf{Y},\ \hat{\mathbf{R}} = \mathbf{R}\mathbf{Y}.$$

The associated $\hat{\mathbf{\Lambda}}$ and $\hat{\mathbf{M}}$ must satisfy the **projected equations** resulting from (9), i.e.

$$\hat{\mathbb{L}_s} - \hat{\mathbb{L}}\,\hat{\mathbf{\Lambda}} = \hat{\mathbf{V}}\hat{\mathbf{R}} \quad \text{and} \quad \hat{\mathbb{L}_s} - \hat{\mathbf{M}}\,\hat{\mathbb{L}} = \hat{\mathbf{L}}\hat{\mathbf{W}} \tag{10}$$

Let the associated EVDs be:

$$\hat{\mathbf{\Lambda}} = \mathbf{T}_\Lambda\mathbf{D}_\Lambda\mathbf{T}_\Lambda^{-1} = \text{eig}\,(\hat{\mathbb{L}_s} - \hat{\mathbf{V}}\hat{\mathbf{R}}, \hat{\mathbb{L}}), \quad \hat{\mathbf{M}} = \mathbf{T}_M\mathbf{D}_M\mathbf{T}_M^{-1} = \text{eig}\,(\hat{\mathbb{L}_s} - \hat{\mathbf{L}}\hat{\mathbf{W}}, \hat{\mathbb{L}}).$$

Consequently, equations (10) are transformed to

$$\bar{\mathbb{L}_s} - \bar{\mathbb{L}}\mathbf{D}_\Lambda = \bar{\mathbf{V}}\bar{\mathbf{R}}, \quad \bar{\mathbb{L}_s} - \mathbf{D}_M\bar{\mathbb{L}} = \bar{\mathbf{L}}\bar{\mathbf{W}} \quad \text{where} \quad \left\{ \begin{array}{llll} \bar{\mathbb{L}_s} = \mathbf{T}_M^{-1}\hat{\mathbb{L}_s}\mathbf{T}_\Lambda, & \bar{\mathbb{L}} = \mathbf{T}_M^{-1}\hat{\mathbb{L}}\mathbf{T}_\Lambda, \\ \bar{\mathbf{V}} = \mathbf{T}_M^{-1}\hat{\mathbf{V}}, & \bar{\mathbf{L}} = \mathbf{T}_M^{-1}\hat{\mathbf{L}}, \\ \bar{\mathbf{W}} = \hat{\mathbf{W}}\,\mathbf{T}_\Lambda, & \bar{\mathbf{R}} = \hat{\mathbf{R}}\,\mathbf{T}_\Lambda. \end{array} \right.$$

Conclusion: for the reduced system the right/left data triples are $(\mathbf{D}_\Lambda, \bar{\mathbf{W}}, \bar{\mathbf{R}})$, $(\mathbf{D}_M, \bar{\mathbf{V}}, \bar{\mathbf{L}})$.

Lemma

1. For $q = k \leq n$, define the transfer function $\hat{\mathbf{H}}(s) = \hat{\mathbf{C}}(s\hat{\mathbf{E}} - \hat{\mathbf{A}})^{-1}\hat{\mathbf{B}}$. The interpolation conditions below are satisfied:

$$\hat{\mathbf{H}}(\mu_i) = \mathbf{H}(\mu_i) \;\text{ and }\; \hat{\mathbf{H}}(\lambda_i) = \mathbf{H}(\lambda_i) \;\text{ for }\; i = 1, \cdots, k. \tag{11}$$

If $k = q = n$, the Loewner quadruple is equivalent to the original quadruple $(\mathbf{C}, \mathbf{E}, \mathbf{A}, \mathbf{B})$.

2. For arbitrary k and q (i.e. $k, q \leq$ or $\geq n$) the Loewner quadruple interpolates the data, even if the pencil $(\mathbb{L}_s, \mathbb{L})$ is singular. This is to be interpreted as follows: that
$$(\mathbb{L}_s - \lambda_i \mathbb{L})\,\mathbf{e}_i = \mathbf{V} \quad \text{and} \quad \mathbf{e}_j^T\,(\mathbb{L}_s - \mu_j \mathbb{L}) = \mathbf{W}.$$
Hence $\mathbf{W}\mathbf{e}_i = \mathbf{w}_i$, $i = 1, \cdots, k$, and $\mathbf{e}_j^T\mathbf{V} = \mathbf{v}_j$, $j = 1, \cdots, q$. Therefore the transfer function of the Loewner pencil interpolates $\mathbf{H}(s)$ at the left and right interpolation points.

3. If $k, \; q \geq n$, let $\boxed{\mathbb{L} = \mathbf{U}_r\boldsymbol{\Sigma}_r\mathbf{V}_r^T, \;\; \mathbf{U}_u \in \mathbb{R}^{q\times r}, \;\; \boldsymbol{\Sigma}_r \in \mathbb{R}^{r\times r}, \;\; \mathbf{V}_r \in \mathbb{R}^{k\times r},}$ be a rank revealing SVD decomposition of $\mathbb{L}$, where $r \leq k, q,$ is the *exact* of the *numerical rank* of $\mathbb{L}$. Let

$$\tilde{\mathbf{E}} = \mathbf{U}_r^T\mathbb{L}\mathbf{V}_r \in \mathbb{C}^{r\times r}, \; \tilde{\mathbf{A}} = \mathbf{U}_r^T\mathbb{L}_s\mathbf{V}_r \in \mathbb{C}^{r\times r}, \; \tilde{\mathbf{B}} = \mathbf{U}_r^T\mathbf{V} \in \mathbb{C}^{r}, \; \tilde{\mathbf{C}} = \mathbf{W}\mathbf{U}_r \in \mathbb{C}^{1\times r}. \tag{12}$$

Then the following approximate interpolation conditions are satisfied:

$$\tilde{\mathbf{H}}(\mu_i) \approx \mathbf{H}(\mu_i), \; i = 1, \cdots, q, \quad \text{and} \quad \tilde{\mathbf{H}}(\lambda_j) \approx \mathbf{H}(\lambda_j), \; j = 1, \cdots, k.$$

Optimal $\mathcal{H}_2$ reduction. The $\mathcal{H}_2$ norm of a stable system $\mathbf{\Sigma} = (\mathbf{C}, \mathbf{E}, \mathbf{A}, \mathbf{B})$, is:

$$\|\mathbf{\Sigma}\|_{\mathcal{H}_2} = \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{trace}\, [\mathbf{H}(i\omega)\mathbf{H}^*(-i\omega)]\, d\omega \right)^{1/2},$$

where $\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}$, is the system transfer function. The goal is to construct a reduced system $\mathbf{\Sigma}_k$ of order k, such that

$$\mathbf{\Sigma}_k = \underset{\deg(\hat{\mathbf{\Sigma}})=k}{\arg\min} \left\| \mathbf{\Sigma} - \hat{\mathbf{\Sigma}} \right\|_{\mathcal{H}_2}.$$

This optimization problem is **nonconvex**. We seek therefore reduced models that satisfy **first-order necessary optimality conditions**. These turn out to be interpolatory conditions. Let the rational function $\mathbf{H}_k$ solve the optimal $\mathcal{H}_2$ problem and let $\hat{\lambda}_i$ denote its poles. Assuming that $m = p = 1$, the following **interpolation conditions** hold:

$$\mathbf{H}(-\hat{\lambda}_i^*) = \mathbf{H}_k(-\hat{\lambda}_i^*) \quad \text{and} \quad \frac{d}{ds}\mathbf{H}(s)\Big|_{s=-\hat{\lambda}_i^*} = \frac{d}{ds}\mathbf{H}_k(s)\Big|_{s=-\hat{\lambda}_i^*}.$$

Thus the (locally) optimal reduced system with transfer function $\mathbf{H}_k$ matches the first two moments of the original system at the **mirror image of its poles**.

IRKA (Beattie-Gugercin-Antoulas). The associated algorithm is as follows. Fix the dimension of the reduced system to $k < n$, and pick the left and right interpolation points to be the same: $\lambda_i = \mu_i \in \mathbb{C}$, $i = 1, \cdots, k$. Then, repeat:

1. Define the generalized controllability and observability matrices $\mathcal{R}$ and $\mathcal{O}$ by means of the scalars $\lambda_1, \cdots, \lambda_k$.
2. Define $\hat{\mathbf{E}}$, $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}$, and $\hat{\mathbf{C}}$ as in the projection lemma above, and compute new scalars $\bar{\lambda}_i$, $i = 1, \cdots, k$, as follows
$$\{\bar{\lambda}_i\} = -\operatorname{eig}(\hat{\mathbf{A}}, \hat{\mathbf{E}}).$$
3. If $\{\lambda\} = \{\bar{\lambda}\}$, the system $(\hat{\mathbf{C}}, \hat{\mathbf{E}}, \hat{\mathbf{A}}, \hat{\mathbf{B}})$ is a (locally) optimal $\mathcal{H}_2$ approximant of the original system. Otherwise go to step 2., and repeat until convergence.

Remark The projectors in the above procedure can also be computed by solving the corresponding Sylvester equations (4).

In the generalization to bilinear systems however, the natural way to compute the projectors is by solving appropriate (bilinear) Sylvester equations. Interpolatory conditions also exist, but are more involved to formulate and deal with.

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The Loewner matrix

Given:

row array $(\mu_j, \mathbf{v}_j),\ \ j = 1, \cdots, q,$

column array $(\lambda_i, \mathbf{w}_i),\ \ i = 1, \cdots, k,$

the associated **Loewner matrix** is:

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1 - \mathbf{w}_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mathbf{v}_1 - \mathbf{w}_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_q - \mathbf{w}_1}{\mu_q - \lambda_1} & \cdots & \frac{\mathbf{v}_q - \mathbf{w}_k}{\mu_q - \lambda_k} \end{bmatrix} \in \mathbb{C}^{q \times k}$$

If $\mathbf{w}_i = \mathbf{g}(\lambda_i)$, $\mathbf{v}_j = \mathbf{g}(\mu_j)$, are **samples** of $\mathbf{g}$:

Main property. Let $\mathbb{L}$ be as above.

Then $k, q \geq \deg \mathbf{g} \ \Rightarrow \ \operatorname{rank} \mathbb{L} = \deg \mathbf{g}$.

Karel Löwner (1893 - 1968)

Ch. Loewner

- Born in Bohemia
- Studied in Prague under Georg Pick
- Emigrated to the US in 1939
- Seminal paper:

Über monotone Matrixfunctionen, Math. Zeitschrift (1934).

Model reduction of descriptor systems

A *descriptor-form representation* is a set of differential and algebraic equations (DAEs):

$$\mathbf{\Sigma}: \quad \mathbf{E}\frac{d}{dt}\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t),$$

$$\text{where} \quad \mathbf{E}, \mathbf{A} \in \mathbb{R}^{n\times n}, \ \ \mathbf{B} \in \mathbb{R}^{n\times m}, \ \ \mathbf{C} \in \mathbb{R}^{p\times n}, \ \ \mathbf{D} \in \mathbb{R}^{p\times m}.$$

Remark. **The D-term**. Consider a rank-revealing factorization

$$\mathbf{D} = \mathbf{D}_1\mathbf{D}_2 \quad \text{where} \quad \mathbf{D}_1 \in \mathbb{R}^{p\times\rho}, \ \mathbf{D}_2 \in \mathbb{R}^{\rho\times m},$$

and $\rho = \operatorname{rank}\mathbf{D}$. It readily follows that:

$$\mathbf{E}_\delta = \begin{bmatrix} \mathbf{E} & \\ & \mathbf{0}_{\rho\times\rho} \end{bmatrix}, \ \mathbf{A}_\delta = \begin{bmatrix} \mathbf{A} & \\ & -\mathbf{I}_\rho \end{bmatrix}, \ \mathbf{B}_\delta = \begin{bmatrix} \mathbf{B} \\ \mathbf{D}_2 \end{bmatrix}, \ \mathbf{C}_\delta = \begin{bmatrix} \mathbf{C} & \mathbf{D}_1 \end{bmatrix},$$

is a descriptor realization of the same system with no **D**-term (i.e. $\mathbf{D}_\delta = \mathbf{0}$).
Reason: the Loewner framework yields precisely such descriptor realizations.

Model reduction: construct reduced-order DAE systems of the form:

$$\hat{\mathbf{\Sigma}}: \ \hat{\mathbf{E}}\frac{d}{dt}\hat{\mathbf{x}}(t) = \hat{\mathbf{A}}\hat{\mathbf{x}}(t) + \hat{\mathbf{B}}\mathbf{u}(t), \ \ \hat{\mathbf{y}}(t) = \hat{\mathbf{C}}\hat{\mathbf{x}}(t) + \hat{\mathbf{D}}\mathbf{u}(t),$$

$$\text{where} \quad \hat{\mathbf{E}}, \hat{\mathbf{A}} \in \mathbb{R}^{r\times r}, \ \ \hat{\mathbf{B}} \in \mathbb{R}^{r\times m}, \ \ \hat{\mathbf{C}} \in \mathbb{R}^{p\times r}, \ \ \hat{\mathbf{D}} \in \mathbb{R}^{p\times m}.$$

Descriptor representation of interpolants and rational approximants

Given: right data: $(\lambda_i; \mathbf{r}_i, \mathbf{w}_i)$, $i = 1, \cdots, k$, and left data: $(\mu_j; \boldsymbol{\ell}_j^*, \mathbf{v}_j^*)$, $j = 1, \cdots, q$.

Problem: Find rational $p \times m$ matrices $\mathbf{H}(s)$, such that:

$$\mathbf{H}(\lambda_i)\mathbf{r}_i = \mathbf{w}_i, \quad \boldsymbol{\ell}_j^* \mathbf{H}(\mu_j) = \mathbf{v}_j^*,$$

where $\mathbf{H}(\lambda_i)$, $\mathbf{H}(\mu_j) \in \mathbb{C}^{p \times m}$, are for instance, S-parameters.

Right data:

$$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{bmatrix} \in \mathbb{C}^{k \times k}, \quad \begin{array}{l} \mathbf{R} = [\mathbf{r}_1 \;\; \mathbf{r}_2, \;\; \cdots \;\; \mathbf{r}_k] \in \mathbb{C}^{m \times k}, \\ \mathbf{W} = [\mathbf{w}_1 \;\; \mathbf{w}_2 \;\; \cdots \;\; \mathbf{w}_k] \in \mathbb{C}^{p \times k}, \end{array}$$

Left data:

$$M = \begin{bmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_q \end{bmatrix} \in \mathbb{C}^{q \times q}, \quad \mathbf{L} = \begin{bmatrix} \boldsymbol{\ell}_1^* \\ \vdots \\ \boldsymbol{\ell}_q^* \end{bmatrix} \in \mathbb{C}^{q \times p}, \quad \mathbf{V} = \begin{bmatrix} \mathbf{v}_1^* \\ \vdots \\ \mathbf{v}_q^* \end{bmatrix} \in \mathbb{C}^{q \times m}$$

▲ A.J. Mayo and A.C. Antoulas, *A framework for the solution of the generalized realization problem*, Linear Algebra and Its Applications, vol. 425, pages 634-662 (2007).

Descriptor representation: the Loewner pencil

Data: $\mathbf{H}(\lambda_i)\mathbf{r}_i = \mathbf{w}_i, \;\; \ell_j\mathbf{H}(\mu_j) = \mathbf{v}_j.$

The **Loewner matrix** $\mathbb{L} \in \mathbb{C}^{q\times k}$ is:

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1\mathbf{r}_1 - \ell_1\mathbf{w}_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mathbf{v}_1\mathbf{r}_k - \ell_1\mathbf{w}_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_q\mathbf{r}_1 - \ell_q\mathbf{w}_1}{\mu_q - \lambda_1} & \cdots & \frac{\mathbf{v}_q\mathbf{r}_k - \ell_q\mathbf{w}_k}{\mu_q - \lambda_k} \end{bmatrix}$$

$\mathbb{L}$ satisfies the Sylvester equation

$$\boxed{\mathbb{L}\Lambda - M\mathbb{L} = \mathbf{VR} - \mathbf{LW}}$$

The **shifted Loewner matrix** $\mathbb{L} \in \mathbb{C}^{q\times k}$ is:

$$\mathbb{L}_s = \begin{bmatrix} \frac{\mu_1\mathbf{v}_1\mathbf{r}_1 - \ell_1\mathbf{w}_1\lambda_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mu_1\mathbf{v}_1\mathbf{r}_k - \ell_1\mathbf{w}_k\lambda_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mu_q\mathbf{v}_q\mathbf{r}_1 - \ell_q\mathbf{w}_1\lambda_1}{\mu_q - \lambda_1} & \cdots & \frac{\mu_q\mathbf{v}_q\mathbf{r}_k - \ell_q\mathbf{w}_k\lambda_k}{\mu_q - \lambda_k} \end{bmatrix}$$

$\mathbb{L}_s$ satisfies the Sylvester equation

$$\boxed{\mathbb{L}_s\Lambda - M\mathbb{L}_s = M\mathbf{VR} - \mathbf{LW}\Lambda}$$

▲ A.C. Antoulas, S. Lefteriu and A.C. Ionita, A Tutorial Introduction to the Loewner Framework for Model Reduction, Chapter 8 in Model Reduction and Approximation, pages 335-376, edited by P. Benner, A. Cohen, M. Ohlberger, and K. Willcox, SIAM, Computational Science and Engineering CS15, (2017).

Construction of Interpolants (Models)

- If the pencil $(\mathbb{L}_s, \mathbb{L})$ is regular, then

$$\mathbf{E} = -\mathbb{L}, \quad \mathbf{A} = -\mathbb{L}_s, \quad \mathbf{B} = \mathbf{V}, \quad \mathbf{C} = \mathbf{W}$$

is a minimal interpolant of the data, i.e., $\mathbf{H}(s)$ interpolates the data:

$$\mathbf{H}(s) = \mathbf{W}(\mathbb{L}_s - s\mathbb{L})^{-1}\mathbf{V}$$

- Otherwise, if the **numerical** $\operatorname{rank}\mathbb{L} = k$, compute the rank revealing *SVD*:

$$\mathbb{L} = \mathbf{Y}\Sigma\mathbf{X}^* \approx \mathbf{Y}_k\Sigma_k\mathbf{X}_k^*$$

Theorem. A realization $[\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}]$, of an approximate interpolant is given as follows:

$$\mathbf{E} = -\mathbf{Y}_k^*\mathbb{L}\mathbf{X}_k, \quad \mathbf{A} = -\mathbf{Y}_k^*\mathbb{L}_s\mathbf{X}_k, \quad \mathbf{B} = \mathbf{Y}_k^*\mathbf{V}, \quad \mathbf{C} = \mathbf{W}\mathbf{X}_k.$$

Remark. ▲ If we have more data than necessary, we can consider $(\mathbb{L}_s, \ \mathbb{L}, \ \mathbf{V}, \ \mathbf{W})$, as a singular model of the data. **Consequence**: The original pencil $(\mathbb{L}_s, \mathbb{L})$ and the projected pencil $(\mathbf{A}, \mathbf{E})$, have the same non-trivial eigenvalues.

▲ A.C. Antoulas, *The Loewner framework and transfer functions of singular/rectangular systems*, Applied Mathematics Letters, vol 54, pages 36-47, 2016.

Proof: the factorization of $\mathbb{L}$, $\mathbb{L}_s$, $\mathbf{V}$, $\mathbf{W}$

▷ Recall the Loewner pencil:

$$(\mathbb{L})_{i,j} = \left[\frac{\mathbf{v}_i - \mathbf{w}_j}{\mu_i - \lambda_j}\right], \quad (\mathbb{L}_s)_{i,j} = \left[\frac{\mu_i \mathbf{v}_i - \lambda_j \mathbf{w}_j}{\mu_i - \lambda_j}\right] \in \mathbb{C}^{q \times k}.$$

▷ Define: $\mathcal{R}$: generalized controllability matrix, $\mathcal{O}$: generalized observability matrix

$$\Downarrow$$

$$\underbrace{\begin{bmatrix} \mathbf{C}(\mu_1 \mathbf{E} - \mathbf{A})^{-1} \\ \vdots \\ \mathbf{C}(\mu_q \mathbf{E} - \mathbf{A})^{-1} \end{bmatrix}}_{\mathcal{O}} \mathbf{E} \underbrace{\begin{bmatrix} (\lambda_1 \mathbf{E} - \mathbf{A})^{-1}\mathbf{B} & \cdots & (\lambda_k \mathbf{E} - \mathbf{A})^{-1}\mathbf{B} \end{bmatrix}}_{\mathcal{R}} = -\mathbb{L} \quad \text{and}$$

$$\underbrace{\begin{bmatrix} \mathbf{C}(\mu_1 \mathbf{E} - \mathbf{A})^{-1} \\ \vdots \\ \mathbf{C}(\mu_q \mathbf{E} - \mathbf{A})^{-1} \end{bmatrix}}_{\mathcal{O}} \mathbf{A} \underbrace{\begin{bmatrix} (\lambda_1 \mathbf{E} - \mathbf{A})^{-1}\mathbf{B} & \cdots & (\lambda_k \mathbf{E} - \mathbf{A})^{-1}\mathbf{B} \end{bmatrix}}_{\mathcal{R}} = -\mathbb{L}_s.$$

Also $\mathbf{V} = \mathbf{C}\mathcal{R}$, $\mathbf{W} = \mathcal{O}\mathbf{B}$.

Proof. **(a)** Multiplying equation the first Sylvester equation by s and subtracting it from equation the second one, we get

$$\mathbf{M}(\mathbb{L}_s - s\mathbb{L}) - (\mathbb{L}_s - s\mathbb{L})\boldsymbol{\Lambda} = (\mathbf{M} - s\mathbf{I})\mathbf{V}\mathbf{R} - \mathbf{L}\mathbf{W}(\boldsymbol{\Lambda} - s\mathbf{I}).$$

Multiplying this equation by $\mathbf{e}_i$ on the right and setting $s = \lambda_i$, we obtain

$$(\mathbf{M} - \lambda_i\mathbf{I})(\mathbb{L}_s - \lambda_i\mathbb{L})\mathbf{e}_i = (\mathbf{M} - \lambda_i\mathbf{I})\mathbf{V}\mathbf{r}_i \;\Rightarrow$$
$$(\mathbb{L}_s - \lambda_i\mathbb{L})\mathbf{e}_i = \mathbf{V}\mathbf{r}_i \;\Rightarrow\; \mathbf{W}\mathbf{e}_i = \mathbf{W}(\mathbb{L}_s - \lambda_i\mathbb{L})^{-1}\mathbf{V}\mathbf{r}_i.$$

Thus $\mathbf{w}_i = \mathbf{H}(\lambda_i)\mathbf{r}_i$.

Next, we multiply the above equation by $\mathbf{e}_j^T$ on the left and set $s = \mu_j$:

$$\mathbf{e}_j^T(\mathbb{L}_s - \mu_j\mathbb{L})(\boldsymbol{\Lambda} - \mu_j\mathbf{I}) = \mathbf{e}_j^T\mathbf{L}\mathbf{W}(\boldsymbol{\Lambda} - \mu_j\mathbf{I}) \;\Rightarrow$$
$$\mathbf{e}_j^T(\mathbb{L}_s - \mu_j\mathbb{L}) = \boldsymbol{\ell}_j\mathbf{W} \;\Rightarrow\; \mathbf{e}_j^T\mathbf{V} = \boldsymbol{\ell}_j^T\mathbf{W}(\mathbb{L}_s - \mu_j\mathbb{L})^{-1}\mathbf{V}.$$

Thus $\mathbf{v}_j^T = \boldsymbol{\ell}_j^T\mathbf{H}(\mu_j)$.

(b) With $\mathbf{K} \in \mathbb{C}^{p\times m}$, the Sylvester equations can be rewritten as

$$\mathbf{M}\mathbb{L} - \mathbb{L}\boldsymbol{\Lambda} = (\mathbf{V} - \mathbf{L}\mathbf{K})\mathbf{R} - \mathbf{L}(\mathbf{W} - \mathbf{K}\mathbf{R}),$$
$$\mathbf{M}(\mathbb{L}_s + \mathbf{L}\mathbf{K}\mathbf{R}) - (\mathbb{L}_s + \mathbf{L}\mathbf{K}\mathbf{R})\boldsymbol{\Lambda} = \mathbf{M}(\mathbf{V} - \mathbf{L}\mathbf{K})\mathbf{R} - \mathbf{L}(\mathbf{W} - \mathbf{K}\mathbf{R})\boldsymbol{\Lambda}.$$

Repeating the procedure with the new quantities the desired result follows:

$$\bar{\mathbb{L}}_s = \mathbb{L}_s + \mathbf{L}\mathbf{K}\mathbf{R}, \quad \bar{\mathbf{V}} = \mathbf{V} - \mathbf{L}\mathbf{K}, \quad \bar{\mathbf{W}} = \mathbf{W} - \mathbf{K}\mathbf{R}.$$

Construction of interpolants: The case of redundant data

As shown in Mayo and A. (2007), the problem has a solution provided that

$$\text{rank}\,[\xi\,\mathbb{L} - \mathbb{L}_s] = \text{rank}\,[\mathbb{L},\ \mathbb{L}_s] = \text{rank}\begin{bmatrix} \mathbb{L} \\ \mathbb{L}_s \end{bmatrix} = r,$$

for all $\xi \in \{\lambda_j\} \cup \{\mu_i\}$.

Consider then, the short SVDs:

$$[\mathbb{L},\ \mathbb{L}_s] = \mathbf{Y}\widehat{\boldsymbol{\Sigma}}_r\tilde{\mathbf{X}}^*, \quad \begin{bmatrix} \mathbb{L} \\ \mathbb{L}_s \end{bmatrix} = \tilde{\mathbf{Y}}\boldsymbol{\Sigma}_r\mathbf{X}^*,$$

where $\widehat{\boldsymbol{\Sigma}}_r,\ \boldsymbol{\Sigma}_r \in \mathbb{R}^{r\times r}$, $\mathbf{Y} \in \mathbb{C}^{q\times r}$, $\mathbf{X} \in \mathbb{C}^{k\times r}$.

Remark. r can be taken as the **numerical rank** of the corresponding quantities.

Theorem. The quadruple $(\mathbf{E}_\delta, \mathbf{A}_\delta, \mathbf{B}_\delta, \mathbf{C}_\delta)$ of size $r \times r$, $r \times r$, $r \times m$, $p \times r$, given by:

$$\mathbf{E}_\delta = -\mathbf{Y}^*\mathbb{L}\mathbf{X}, \quad \mathbf{A}_\delta = -\mathbf{Y}^*\mathbb{L}_s\mathbf{X}, \quad \mathbf{B}_\delta = \mathbf{Y}^*\mathbf{V}, \quad \mathbf{C}_\delta = \mathbf{W}\mathbf{X},$$

is a descriptor realization of an (approximate) interpolant of the data with McMillan degree $r = \operatorname{rank} \mathbb{L}$.

Remarks. (a) The Loewner approach constructs a descriptor representation

$$(\mathbb{L},\ \mathbb{L}_s,\ \mathbf{V},\ \mathbf{W}),$$

of an underlying dynamical system exclusively from the data, with no further manipulations involved (i.e. matrix factorizations or inversions). In general, the pencil $(\mathbb{L}_s, \mathbb{L})$ is singular and needs to be projected to a regular pencil $(\mathbf{A}_\delta, \mathbf{E}_\delta)$.

(b) In the Loewner framework, by construction, $\mathbf{D}$ terms are absorbed in the other matrices of the realization. Extracting the $\mathbf{D}$ term involves an eigenvalue decomposition of $(\mathbb{L}_s,\ \mathbb{L})$.

Construction of Interpolants (Models)

- If the pencil $(\mathbb{L}_s, \mathbb{L})$ is regular, i.e. $\mathbf{\Phi}(s) = \mathbb{L}_s - s\,\mathbb{L}$, is invertible, then

$\mathbf{E} = -\mathbb{L}, \;\; \mathbf{A} = -\mathbb{L}_s, \;\; \mathbf{B} = \mathbf{V}, \;\; \mathbf{C} = \mathbf{W}$
is a minimal interpolant of the data $\Rightarrow$ $\boxed{\mathbf{H}(s) = \mathbf{W}\,\mathbf{\Phi}(s)^{-1}\,\mathbf{V}}$

- If $\mathbf{\Phi}(s) = \mathbb{L}_s - s\,\mathbb{L}$, is singular, let $\mathbf{\Phi}(s)^{\#}$ be a generalized inverse of $\mathbf{\Phi}(s)$ (Drazin or Moore-Penrose). $\Rightarrow$ $\boxed{\mathbf{H}(s) = \mathbf{W}\,\mathbf{\Phi}(s)^{\#}\,\mathbf{V}}$

- In the latter case, if the **numerical** $\mathrm{rank}\,\mathbb{L} = k$, compute the rank revealing *SVD*:

$$\boxed{\mathbb{L} = \mathbf{Y}\mathbf{\Sigma}\mathbf{X}^* \approx \mathbf{Y}_k\mathbf{\Sigma}_k\mathbf{X}_k^*}$$

Theorem. A realization $[\mathbf{C}, \mathbf{E}, \mathbf{A}, \mathbf{B}]$, of an approximate interpolant is given as follows:

$$\mathbf{E} = -\mathbf{Y}_k^*\mathbb{L}\mathbf{X}_k, \quad \mathbf{A} = -\mathbf{Y}_k^*\mathbb{L}_s\mathbf{X}_k, \quad \mathbf{B} = \mathbf{Y}_k^*\mathbf{V}, \quad \mathbf{C} = \mathbf{W}\mathbf{X}_k.$$

A simple example

Consider the system

$$\begin{array}{rcl}\dot{\mathbf{x}}_1(t) & = & \mathbf{x}_2(t), \\ \dot{\mathbf{x}}_2(t) & = & -\mathbf{x}_1(t) - \mathbf{x}_2(t) + \mathbf{u}(t)\end{array}, \quad \mathbf{y}(t) = \mathbf{x}_2(t) \quad \Rightarrow \quad \mathbf{H}(s) = \frac{s}{s^2+s+1}.$$

We now wish to recover state equations equivalent to the ones above from measurements of the transfer function.

Data: obtained by evaluating the transfer function at $\lambda_1 = \frac{1}{2}$, $\lambda_2 = 1$, as well as $\mu_1 = -\frac{1}{2}$, $\mu_2 = -1$. The corresponding values of $\mathbf{H}$ are collected in the matrices

$$\mathbf{W} = \left(\begin{array}{cc} \frac{2}{7} & \frac{1}{3} \end{array} \right), \quad \mathbf{V} = \left(\begin{array}{cc} -\frac{2}{3} & -1 \end{array} \right)^T.$$

Furthermore with $\mathbf{R} = [1 \;\; 1]$, $\mathbf{L} = \mathbf{R}^T$, we construct the *Loewner pencil*:

$$\mathbb{L} = \begin{bmatrix} \frac{20}{21} & \frac{2}{3} \\ \frac{6}{7} & \frac{2}{3} \end{bmatrix}, \quad \mathbb{L}_s = \begin{bmatrix} -\frac{4}{21} & 0 \\ -\frac{4}{7} & -\frac{1}{3} \end{bmatrix}.$$

Since the pencil $(\mathbb{L}_s, \; \mathbb{L})$ is regular, and the rank of both matrices is two:

$$\mathbf{H}(s) = \mathbf{W}\mathbf{\Phi}(s)^{-1}\mathbf{V} = \frac{s}{s^2+s+1}, \quad \text{where} \quad \mathbf{\Phi}(s) = \mathbb{L}_s - s\,\mathbb{L}.$$

Hence, the measurements above yield a minimal (descriptor) realization of the system in terms of the (state) variables ξ_1, ξ_2:

$$
\begin{aligned}
\tfrac{20}{21}\,\dot{\xi}_1(t) + \tfrac{2}{3}\,\dot{\xi}_1(t) &= -\tfrac{4}{21}\xi_1(t) + \tfrac{2}{3}\mathbf{u}(t),\\
\tfrac{6}{7}\,\dot{\xi}_1(t) + \tfrac{2}{3}\,\dot{\xi}_2(t) &= -\tfrac{4}{7}\,\xi_1(t) - \tfrac{1}{3}\,\xi_2(t) + \mathbf{u}(t),\\
\mathbf{y}(t) &= \tfrac{2}{7}\,\xi_1(t) + \tfrac{1}{3}\,\xi_2(t).
\end{aligned}
$$

Question: what happens if we collect more data that necessary:

$$
\boldsymbol{\Lambda} = \mathrm{diag}\left(\begin{array}{cccc} \frac{1}{2} & 1 & \frac{3}{2} & 2 \end{array}\right),\;\; \mathbf{M} = \mathrm{diag}\left(\begin{array}{cccc} -\frac{1}{2} & -1 & -\frac{3}{2} & -2 \end{array}\right).
$$

In this case, the associated measurements are

$$
\mathbf{W} = \left(\begin{array}{cccc} \frac{2}{7} & \frac{1}{3} & \frac{6}{19} & \frac{2}{7} \end{array}\right),\;\; \mathbf{V} = \left(\begin{array}{cccc} -\frac{2}{3} & -1 & -\frac{6}{7} & -\frac{2}{3} \end{array}\right)^T,
$$

and with $\mathbf{R} = [1 \;\; 1 \;\; 1 \;\; 1]$, $\mathbf{L} = \mathbf{R}^T$, the Loewner pencil is:

$$
\mathbb{L} = \left[\begin{array}{cc|cc}
\frac{20}{21} & \frac{2}{3} & \frac{28}{57} & \frac{8}{21}\\
\frac{6}{7} & \frac{2}{3} & \frac{10}{19} & \frac{3}{7}\\ \hline
\frac{4}{7} & \frac{10}{21} & \frac{52}{133} & \frac{16}{49}\\
\frac{8}{21} & \frac{1}{3} & \frac{16}{57} & \frac{5}{21}
\end{array}\right],\;\;
\mathbb{L}_s = \left[\begin{array}{cc|cc}
-\frac{4}{21} & 0 & \frac{4}{57} & \frac{2}{21}\\
-\frac{4}{7} & -\frac{1}{3} & -\frac{4}{19} & -\frac{1}{7}\\ \hline
-\frac{4}{7} & -\frac{8}{21} & -\frac{36}{133} & -\frac{10}{49}\\
-\frac{10}{21} & -\frac{1}{3} & -\frac{14}{57} & -\frac{4}{21}
\end{array}\right].
$$

It turns out that we can choose **arbitrary** $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{4\times 2}$, such that $\mathbf{Y}^T\mathbf{X}$ is nonsingular, e.g.

$$\mathbf{X} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \\ -2 & 1 \end{bmatrix}, \quad \mathbf{Y}^T = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix},$$

so that the projected quantities

$$\widehat{\mathbf{W}} = \mathbf{W}\mathbf{X} = \begin{bmatrix} -\frac{6}{7} & -\frac{1}{21} \end{bmatrix}, \quad \widehat{\mathbb{L}} = \mathbf{Y}^T\mathbb{L}\mathbf{X} = \begin{bmatrix} -\frac{6}{7} & -\frac{1}{7} \\ \frac{18}{49} & \frac{1}{147} \end{bmatrix},$$

$$\widehat{\mathbb{L}}_s = \mathbf{Y}^T\mathbb{L}_s\mathbf{X} = \begin{bmatrix} 0 & \frac{1}{21} \\ -\frac{48}{49} & -\frac{19}{147} \end{bmatrix}, \quad \widehat{\mathbf{V}} = \mathbf{Y}^T\mathbf{V} = \begin{bmatrix} -\frac{1}{3} \\ \frac{11}{21} \end{bmatrix},$$

constitute a minimal realization of $\mathbf{H}(s)$:

$$\boxed{\mathbf{H}(s) = \widehat{\mathbf{W}}\left(\widehat{\mathbb{L}}_s - s\widehat{\mathbb{L}}\right)^{-1}\widehat{\mathbf{V}}} = \frac{s}{s^2+s+1}.$$

Here we would like to recover the rational function $\mathbf{H}(s) = \frac{1}{s^2+1}$, from the following measurements: $\lambda_1 = 1$; $\lambda_2 = 2$; $\lambda_3 = 3$; $\mu_1 = -1$; $\mu_2 = -2$; $\mu_3 = -3$; $\mathbf{W} = [\frac{1}{2}, \quad \frac{1}{5}, \quad \frac{1}{10}] = \mathbf{V}^T$; $\mathbf{R} = [1 \;\; 1 \;\; 1] = \mathbf{L}^T$; it follows that

$$\mathbb{L} = \begin{bmatrix} 0 & \frac{-1}{10} & \frac{-1}{10} \\ \frac{1}{10} & 0 & \frac{-1}{50} \\ \frac{1}{10} & \frac{1}{50} & 0 \end{bmatrix}, \quad \mathbb{L}_s = \begin{bmatrix} \frac{1}{2} & \frac{3}{10} & \frac{1}{5} \\ \frac{3}{10} & \frac{1}{5} & \frac{7}{50} \\ \frac{1}{5} & \frac{7}{50} & \frac{1}{10} \end{bmatrix}.$$

We choose the following projection matrices

$$\mathbf{X} = \begin{bmatrix} 5 & -5 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{Y} = \mathbf{X}^T$$

Thus the projected quantites are

$$\mathbb{L}_0 = \mathbf{Y}\mathbb{L}\mathbf{X} = \begin{bmatrix} 0 & -\frac{51}{50} \\ \frac{51}{50} & 0 \end{bmatrix}, \quad \mathbb{L}_{s0} = \mathbf{Y}\mathbb{L}_s\mathbf{X} = \begin{bmatrix} \frac{157}{10} & -\frac{643}{50} \\ -\frac{643}{50} & \frac{53}{5} \end{bmatrix},$$

$$\mathbf{W}_0 = \mathbf{W}\mathbf{X} = \left[\frac{27}{10}, -\frac{12}{5}\right], \mathbf{V}_0 = \mathbf{Y}\mathbf{V} = \begin{bmatrix} \frac{27}{10} \\ -\frac{12}{5} \end{bmatrix}, \mathbf{R}_0 = \mathbf{R}\mathbf{X} = [6, \quad -4], \mathbf{L}_0 = \mathbf{Y}\mathbf{L} = \begin{bmatrix} 6 \\ -4 \end{bmatrix}.$$

Therefore it readily follows that we recover the original rational function:

$$(\mathbf{W}\mathbf{X})\left[(\mathbf{Y}\mathbb{L}_s\mathbf{X}) - s(\mathbf{Y}\mathbb{L}\mathbf{X})\right]^{-1}(\mathbf{Y}\mathbf{V}) = \frac{1}{s^2+1}.$$

Example. Next, we wish to recover the polynomial $\phi(s) = s^2$, my means of measurements. From $\mathbf{\Lambda} = \text{diag}\,(\ 1,\ 2,\ 3)$, $\mathbf{M} = \text{diag}\,(\ -1,\ -2,\ -3)$, $\mathbf{W} = \mathbf{V}^* = [1,\ \ 4,\ \ 9]$, we calculate

$$\mathbb{L} = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & -1 & 0 \end{bmatrix}, \quad \mathbb{L}_s = \begin{bmatrix} 1 & 3 & 7 \\ 3 & 4 & 7 \\ 7 & 7 & 9 \end{bmatrix}.$$

Since rank $(\mathbb{L}) = 2$, but rank $(\mathbb{L}_s) = 3$, the McMillan degree of the minimal interpolant is 2. Therefore $\mathbf{A} = -\mathbb{L}_s$, $\mathbf{E} = -\mathbb{L}$, $\mathbf{B} = \mathbf{V}$, $\mathbf{C} = \mathbf{W}$ is a descriptor realization of the McMillan degree 2, interpolant: $\phi(s) = \mathbf{W}(\mathbb{L}_s - s\,\mathbb{L})^{-1}\mathbf{V}$.

We now consider two additional points: $\mathbf{\Lambda} = \text{diag}\,[1,\ \ 2,\ \ 3,\ \ 4]$, $\mathbf{M} = -\Lambda$. Then $\mathbf{W} = \mathbf{V}^* = [1,\ \ 4,\ \ 9,\ \ 16]$. Thus the Loewner pencil is updated by means of a new row and a new column:

$$\mathbb{L} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ -1 & 0 & 1 & 2 \\ -2 & -1 & 0 & 1 \\ -3 & -2 & -1 & 0 \end{bmatrix}, \quad \mathbb{L} = \begin{bmatrix} 1 & 3 & 7 & 13 \\ 3 & 4 & 7 & 12 \\ 7 & 7 & 9 & 13 \\ 13 & 12 & 13 & 16 \end{bmatrix}.$$

Indeed the pencil $(\mathbb{L}_s, \mathbb{L})$ has a (generalized) eigenvalue at 0 and a corresponding Jordan chain of length 3:

$$\mathbf{v}_0 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -2 \\ 4 \\ 0 \\ 1 \end{bmatrix},$$

satisfying: $\mathbb{L}\mathbf{v}_0 = \mathbf{0}$, $\mathbb{L}_s\mathbf{v}_0 = \mathbb{L}\mathbf{v}_1$, $\mathbb{L}_s\mathbf{v}_1 = \mathbb{L}\mathbf{v}_2$.

Alternatively one may compute the QZ factorization of $(\mathbb{L}_s, \mathbb{L})$, to obtain an upper triangular pencil with diagonal entries:

$$\begin{array}{ll|ll} -3.3467\,10^{-8} & -1.8733\,10^{-1}i & 1.4534\,10^{-8} & \leftarrow \text{ zero eig} \\ -2.0228\,10^{-7} & +1.1323i & 8.7842\,10^{-8} & \leftarrow \text{ zero eig} \\ \ \ 5.3653\,10^{-15} & & 1.5794\,10^{-15} & \\ \ \ 3.9262 & & 0 & \leftarrow \text{ zero eig} \end{array}$$

Consequently, the quotients of the first, second and fourth entries of the diagonal yield the three zero eigenvalues, while the third diagonals indicate an undetermined eigenvalue.

We will now project the quadruple $(\mathbb{L}_s, \mathbb{L}, \mathbf{V}, \mathbf{W})$ to get a minimal realization. The projectors are chosen randomly (use command round(randn(4,3)) in `Matlab`):

$$\mathbf{T}_1 = \begin{bmatrix} -2 & 1 & 0 \\ -1 & 0 & -1 \\ 1 & 1 & 3 \\ -1 & -2 & 1 \end{bmatrix}, \mathbf{T}_2 = \begin{bmatrix} 1 & 0 & 1 & -1 \\ -1 & 1 & -2 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \det(\mathbf{T}_2\mathbf{T}_1) = 30 \neq 0,$$

which shows that the condition of corollary **??** is satisfied. Consequently, the projected quadruple yields a minimal realization of the underlying rational function:

$$\mathbf{T}_2\mathbb{L}\mathbf{T}_1 = \begin{bmatrix} -5 & -4 & 11 \\ -19 & 16 & -5 \\ 7 & -4 & -1 \end{bmatrix}, \quad \mathbf{T}_2\mathbb{L}_s\mathbf{T}_1 = \begin{bmatrix} 5 & -22 & 21 \\ 128 & 36 & -154 \\ -41 & -6 & 43 \end{bmatrix},$$

and hence $[\mathbf{W}\mathbf{T}_1] \cdot [(\mathbf{T}_2\mathbb{L}_s\mathbf{T}_1) - s \cdot (\mathbf{T}_2\mathbb{L}\mathbf{T}_1)]^{-1} \cdot [\mathbf{T}_2\mathbf{V}] = s^2 = \phi(s)$. ■

There is another way to express the above relationship avoiding arbitrary projectors. Basic ingredients: the **Moore-Penrose generalized inverse** and the **Drazin generalized inverse**.

The Moore-Penrose inverse of the (rectangular) matrix $\mathbf{M} \in \mathbb{R}^{q\times k}$, is denoted by $\mathbf{M}^{MP} \in \mathbb{R}^{k\times q}$, and satisfies: (a) $\mathbf{M}\mathbf{M}^{MP}\mathbf{M} = \mathbf{M}$, (b) $\mathbf{M}^{MP}\mathbf{M}\mathbf{M}^{MP} = \mathbf{M}^{MP}$, (c) $\left[\mathbf{M}\mathbf{M}^{MP}\right]^T = \mathbf{M}\mathbf{M}^{MP}$, (d) $\left[\mathbf{M}^{MP}\mathbf{M}\right]^T = \mathbf{M}^{MP}\mathbf{M}$.

This generalized inverse always exists and is unique.

Given a square matrix $\mathbf{M} \in \mathbb{R}^{q\times q}$, its *index* is the least nonnegative integer κ such that $\operatorname{rank}\mathbf{M}^{\kappa+1} = \operatorname{rank}\mathbf{M}^{\kappa}$.
The Drazin inverse of $\mathbf{M}$ is the unique matrix $\mathbf{M}^D$ satisfying:
(a) $\mathbf{M}^{\kappa+1}\mathbf{M}^D = \mathbf{M}^{\kappa}$, (b) $\mathbf{M}^D\mathbf{M}\mathbf{M}^D = \mathbf{M}^D$,
(c) $\mathbf{M}\mathbf{M}^D = \mathbf{M}^D\mathbf{M}$.

In the sequel we will be concerned with rectangular $n \times m$ *polynomial matrices* which have an explicit (rank revealing) factorization as follows:

$$\mathbf{M} = \mathbf{X}\boldsymbol{\Delta}\mathbf{Y}^T,$$

where $\mathbf{X}$, $\boldsymbol{\Delta}$, $\mathbf{Y}$ have dimension $q \times n$, $n \times n$, $n \times k$, $n \le q, k$, and all have full rank k.

The **Moore-Penrose** generalized inverse is:

$$\mathbf{M}^{MP} = \mathbf{Y}(\mathbf{Y}^T\mathbf{Y})^{-1}\boldsymbol{\Delta}^{-1}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T.$$

If $q = k$ and $\mathbf{Y}^T\mathbf{X}$ is *invertible*, the **Drazin generalized inverse** is:

$$\mathbf{M}^D = \mathbf{X}(\mathbf{Y}^T\mathbf{X})^{-1}\boldsymbol{\Delta}^{-1}(\mathbf{Y}^T\mathbf{X})^{-1}\mathbf{Y}^T.$$

A.C. Antoulas, *The Loewner framework and transfer functions of singular/rectangular systems*, Applied Mathematics Letters, **54**: 36-47 (2016).

Example (continued). The quantities needed are the generalized inverses of

$$\mathbf{\Phi}(s) = \mathbb{L}_s - s\,\mathbb{L} = \begin{bmatrix} -\frac{20s}{21} - \frac{4}{21} & -\frac{2s}{3} & \frac{4}{57} - \frac{28s}{57} & \frac{2}{21} - \frac{8s}{21} \\ -\frac{6s}{7} - \frac{4}{7} & -\frac{2s}{3} - \frac{1}{3} & -\frac{10s}{19} - \frac{4}{19} & -\frac{3s}{7} - \frac{1}{7} \\ -\frac{4s}{7} - \frac{4}{7} & -\frac{10s}{21} - \frac{8}{21} & -\frac{52s}{133} - \frac{36}{133} & -\frac{16s}{49} - \frac{10}{49} \\ -\frac{8s}{21} - \frac{10}{21} & -\frac{s}{3} - \frac{1}{3} & -\frac{16s}{57} - \frac{14}{57} & -\frac{5s}{21} - \frac{4}{21} \end{bmatrix} = \mathbf{X}\mathbf{\Delta}(s)\mathbf{Y}^T.$$

Let the common range of the columns of $\mathbb{L}$, $\mathbb{L}_s$ be spanned by the columns of $\mathbf{X}$ and the common range of the rows of the same matrices by the rows of $\mathbf{Y}$; it follows that

$$\mathbf{X} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -\frac{3}{7} & \frac{8}{7} \\ -\frac{1}{2} & 1 \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} 1 & 0 & -\frac{7}{19} & -\frac{1}{2} \\ 0 & 1 & \frac{24}{19} & \frac{9}{7} \end{bmatrix} \quad \Rightarrow \quad \det(\mathbf{YX}) \neq 0.$$

Thus with $\mathbf{\Delta}(s) = \mathbf{\Phi}(1:2, 1:2)(s)$ there holds

$$\mathbf{\Phi}(s)^{MP} = \frac{1}{80989667}\,\frac{1}{s^2+s+1}\,\cdot$$

$$\cdot \begin{bmatrix} -28\,(11610185s + 7274073) & 14\,(3558666s - 5604037) & 6076\,(32301s - 391) & 14\,(15168851s + 1670036) \\ 294\,(225182s + 281171) & (-147)\,(192415s - 19668) & -2058\,(29494s + 15609) & -147\,(417597s + 261503) \\ 3724\,(54617s + 48189) & (-1862)\,(29046s - 17485) & -26068\,(5715s + 1523) & -1862\,(83663s + 30704) \\ 98\,(2527157s + 2123670) & -49\,(1250553s - 876439) & -98\,(1797669s + 409322) & -49\,(3777710s + 1247231]) \end{bmatrix}$$

and $\quad \mathbf{\Phi}(s)^D = \frac{1}{4897369}\frac{1}{s^2+s+1}\cdot$

$$\cdot \begin{bmatrix} -84\,(234677s+152881) & 294\,(10652s-13755) & 588\,(19079s-641) & 42\,(330545s+29086) \\ 126\,(31956s+42829) & -147\,(11885s+4) & -882\,(4184s+2255) & -63\,(67611s+42841) \\ 684\,(19079s+17063) & -798\,(4184s-2171) & -4788\,(1885s+441) & -342\,(31631s+10550) \\ 42\,(330545s+281368) & -147\,(22537s-13751) & -294\,(31631s+6124) & -21\,(533378s+157609) \end{bmatrix}.$$

In the **rectangular case**, where there are two less right measuremnents, i.e we only have $\widetilde{\mathbf{\Lambda}} = \text{diag}\left[\frac{1}{2}, \quad 1\right]$, while $\mathbf{M}$ remains the same, the right values are $\widetilde{\mathbf{W}} = \mathbf{W}(:, 1:2)$; hence

$$\widetilde{\mathbf{\Phi}}(s) = \widetilde{\mathbb{L}}_s - s\,\widetilde{\mathbb{L}} = \left[\begin{array}{c|c} -\frac{20s}{21}-\frac{4}{21} & -\frac{2s}{3} \\ -\frac{6s}{7}-\frac{4}{7} & -\frac{2s}{3}-\frac{1}{3} \\ -\frac{4s}{7}-\frac{4}{7} & -\frac{10s}{21}-\frac{8}{21} \\ -\frac{8s}{21}-\frac{10}{21} & -\frac{s}{3}-\frac{1}{3} \end{array}\right] = \mathbf{X}\,\mathbf{\Delta}(s)\,\widetilde{\mathbf{Y}}^T,$$

has dimension 4×2, where $\widetilde{\mathbf{Y}} = \mathbf{Y}(1:2, 1:2)$. In this case the Moore-Penrose inverse is

$$\widetilde{\mathbf{\Phi}}(s)^{MP} = \frac{1}{737\,(s^2+s+1)}\left[\begin{array}{cccc} -4767s-3402 & \frac{1827}{2}s-\frac{2037}{2} & 3087s+294 & 3297s+\frac{1365}{2} \\ \hline 5838s+5250 & -1596s+903 & -4326s-1218 & -4515s-1722 \end{array}\right].$$

$$\Rightarrow \quad \boxed{\mathbf{W}\,\mathbf{\Phi}(s)^{MP}\,\mathbf{V} = \widetilde{\mathbf{W}}\,\widetilde{\mathbf{\Phi}}(s)^{MP}\,\mathbf{V} = \mathbf{W}\,\mathbf{\Phi}(s)^{D}\,\mathbf{V} = \mathbf{H}(s)}$$

Thus, the Loewner framework allows the definition of **rectangular and/or singular** systems.

Revisit: Construction of Interpolants

- If the pencil $(\mathbb{L}_s, \mathbb{L})$ is regular, i.e. $\boldsymbol{\Phi}(s) = \mathbb{L}_s - s\,\mathbb{L}$, is invertible, then

$\mathbf{E} = -\mathbb{L}, \;\; \mathbf{A} = -\mathbb{L}_s, \;\; \mathbf{B} = \mathbf{V}, \;\; \mathbf{C} = \mathbf{W}$
is a minimal interpolant of the data $\Rightarrow$ $\boxed{\mathbf{H}(s) = \mathbf{W}\,\boldsymbol{\Phi}(s)^{-1}\,\mathbf{V}}$

- If $\boldsymbol{\Phi}(s) = \mathbb{L}_s - s\,\mathbb{L}$, is singular, let $\boldsymbol{\Phi}(s)^{\#}$ be a generalized inverse of $\boldsymbol{\Phi}(s)$ (Drazin or Moore-Penrose). $\Rightarrow$ $\boxed{\mathbf{H}(s) = \mathbf{W}\,\boldsymbol{\Phi}(s)^{\#}\,\mathbf{V}}$

- In the latter case, if the **numerical** $\operatorname{rank}\mathbb{L} = k$, compute the rank revealing *SVD*:

$$\boxed{\mathbb{L} = \mathbf{Y}\boldsymbol{\Sigma}\mathbf{X}^* \approx \mathbf{Y}_k\boldsymbol{\Sigma}_k\mathbf{X}_k^*}$$

Theorem. A realization $[\mathbf{C}, \mathbf{E}, \mathbf{A}, \mathbf{B}]$, of an approximate interpolant is given as follows:

$$\mathbf{E} = -\mathbf{Y}_k^*\mathbb{L}\mathbf{X}_k, \quad \mathbf{A} = -\mathbf{Y}_k^*\mathbb{L}_s\mathbf{X}_k, \quad \mathbf{B} = \mathbf{Y}_k^*\mathbf{V}, \quad \mathbf{C} = \mathbf{W}\mathbf{X}_k.$$

The Loewner Algorithm (simple version)

1. Consider given (frequency domain) measurements (s_i, ϕ_i), $i = 1, \dots, N$.

2. Partition the measurements into 2 disjoint sets

$$\begin{aligned} \text{frequencies}: \quad [s_1, \cdots, s_N] &= [\lambda_1, \cdots, \lambda_k], \; [\mu_1, \cdots, \mu_q], \quad k + q = N, \\ \text{values}: \quad [\phi_1, \cdots, \phi_N] &= [w_1, \cdots, w_k], \; [v_1, \cdots v_q] \; = \mathbf{W}, \; \mathbf{V}^T. \end{aligned}$$

3. Construct the **Loewner pencil**:

$$\mathbb{L} = \left(\frac{v_i - w_j}{\mu_i - \lambda_j} \right)_{i=1,\cdots,q}^{j=1,\cdots,k}, \qquad \mathbb{L}_s = \left(\frac{\mu_i v_i - \lambda_j w_j}{\mu_i - \lambda_j} \right)_{i=1,\cdots,q}^{j=1,\cdots,k}.$$

4. It follows that the **raw model** is: $(\mathbf{W}, \mathbb{L}, \mathbb{L}_s, \mathbf{V})$.

5. Compute the rank revealing SVD: $\mathbb{L} \approx \mathbf{Y \Sigma X}^*$ $(\Sigma \in \mathbb{R}^{r \times r})$.

6. The reduced model $(\widehat{\mathbf{C}}, \widehat{\mathbf{E}}, \widehat{\mathbf{A}}, \widehat{\mathbf{B}})$ is obtained by **projecting** the raw model $(\mathbf{W}, \mathbb{L}, \mathbb{L}_s, \mathbf{V})$:

$$\widehat{\mathbf{C}} = \mathbf{WX}, \quad \widehat{\mathbf{E}} = -\mathbf{Y}^* \mathbb{L} \mathbf{X}, \quad \widehat{\mathbf{A}} = -\mathbf{Y}^* \mathbb{L}_s \mathbf{X}, \quad \widehat{\mathbf{B}} = \mathbf{Y}^* \mathbf{V}.$$

7. Reference: S. Lefteriu and A.C. Antoulas: A New Approach to Modeling Multiport Systems from Frequency-Domain Data, IEEE Trans. CAD, 29: 14-27 (2010).

An illustrative example

Illustration of the *relationship between the McMillan degree, the degree of minimal realizations, and the* **D***-term*.

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},\ \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},\ \mathbf{B} = \mathbf{I}_3,\ \mathbf{D} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \ \Rightarrow$$

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} = \begin{bmatrix} \frac{1}{s}+1 & \frac{1}{s^2}+1 & \frac{1}{s^3}+1 \\ 1 & \frac{1}{s}+1 & \frac{1}{s^2}+1 \end{bmatrix}.$$

Let the interpolation points be:

$$\mathbf{M} = \operatorname{diag}\left[1,\ 1,\ -\tfrac{1}{2},\ -\tfrac{1}{2},\ -\tfrac{1}{4},\ -\tfrac{1}{4}\right],$$
$$\mathbf{\Lambda} = \operatorname{diag}\left[\tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2},\ -1,\ -1,\ -1,\ 2,\ 2,\ 2\right].$$

The interpolation values $\mathbf{w}_i = \mathbf{H}(\lambda_i)$, $\mathbf{v}_i = \mathbf{H}(\mu_i)$, $i = 1, 2, 3$, are:

$$\mathbf{w}_1 = \begin{bmatrix} 3 & 5 & 9 \\ 1 & 3 & 5 \end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}, \mathbf{w}_3 = \begin{bmatrix} 3/2 & 5/4 & 9/8 \\ 1 & 3/2 & 5/4 \end{bmatrix},$$

$$\mathbf{v}_1 = \begin{bmatrix} 2 & 2 & 2 \\ 1 & 2 & 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 & 5 & -7 \\ 1 & -1 & 5 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -3 & 17 & -63 \\ 1 & -3 & 17 \end{bmatrix}.$$

This corresponds to *matrix interpolation* (the values considered are matrices) as opposed to tangential interpolation (which will follow next)

$$\mathbf{V} = \left[\begin{array}{rrr} 2 & 2 & 2 \\ 1 & 2 & 2 \\ \hline -1 & 5 & -7 \\ 1 & -1 & 5 \\ \hline -3 & 17 & -63 \\ 1 & -3 & 17 \end{array}\right], \quad \mathbf{W} = \left[\begin{array}{ccc|ccc|ccc} 3 & 5 & 9 & 0 & 2 & 0 & \frac{3}{2} & \frac{5}{4} & \frac{9}{8} \\ 1 & 3 & 5 & 1 & 0 & 2 & 1 & \frac{3}{2} & \frac{5}{4} \end{array}\right],$$

$$\mathbf{R} = [\mathbf{I}_3 \;\; \mathbf{I}_3 \;\; \mathbf{I}_3], \quad \mathbf{L} = \left[\begin{array}{c} \mathbf{I}_2 \\ \mathbf{I}_2 \\ \mathbf{I}_2 \end{array}\right].$$

$$\Rightarrow \quad \mathbb{L} = \left[\begin{array}{rrr|rrr|rrr} -2 & -6 & -14 & 1 & 0 & 1 & -\frac{1}{2} & -\frac{3}{4} & -\frac{7}{8} \\ 0 & -2 & -6 & 0 & 1 & 0 & 0 & -\frac{1}{2} & -\frac{3}{4} \\ \hline 4 & 0 & 16 & -2 & 6 & -14 & 1 & -\frac{3}{2} & \frac{13}{4} \\ 0 & 4 & 0 & 0 & -2 & 6 & 0 & 1 & -\frac{3}{2} \\ \hline 8 & -16 & 96 & -4 & 20 & -84 & 2 & -7 & \frac{57}{2} \\ 0 & 8 & -16 & 0 & -4 & 20 & 0 & 2 & -7 \end{array}\right] \in \mathbb{R}^{6\times 9},$$

$$\text{and} \qquad \mathbb{L}_s = \left[\begin{array}{rrr|rrr|rrr} 1 & -1 & -5 & 1 & 2 & 1 & 1 & \frac{1}{2} & \frac{1}{4} \\ 1 & 1 & -1 & 1 & 1 & 2 & 1 & 1 & \frac{1}{2} \\ \hline 1 & 5 & 1 & 1 & -1 & 7 & 1 & 2 & -\frac{1}{2} \\ 1 & 1 & 5 & 1 & 1 & -1 & 1 & 1 & 2 \\ \hline 1 & 9 & -15 & 1 & -3 & 21 & 1 & 3 & -6 \\ 1 & 1 & 9 & 1 & 1 & -3 & 1 & 1 & 3 \end{array}\right] \in \mathbb{R}^{6\times 9}.$$

It readily follows that, while $\operatorname{rank}\mathbb{L} = \operatorname{rank}\mathbb{L}_s = 3$, the rank of $[\mathbb{L};\ \mathbb{L}_s]$ is equal to the rank of $[\mathbb{L},\ \mathbb{L}_s]$, which is equal to 4. Furthermore the rank of $\xi\mathbb{L} - \mathbb{L}_s$ is also 4, for all $\xi \in \{\mu_i\} \cup \{\lambda_j\}$. Consequently the dimension of the minimal realization is $r = 4$.

Tangential interpolation. If we define the index set $I = [1\ 2\ 3\ 4]$, we get:

$$\mathbf{\Lambda}(I,I) = \text{diag}\left[\begin{array}{cccc} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -1 \end{array}\right], \ \mathbf{M}(I,I) = \text{diag}\left[\begin{array}{cccc} 1 & 1 & -\frac{1}{2} & -\frac{1}{2} \end{array}\right],$$

$$\mathbf{W}(:,I) = \left[\begin{array}{cccc} 3 & 5 & 9 & 0 \\ 1 & 3 & 5 & 1 \end{array}\right], \ \mathbf{V}(I,:) = \left[\begin{array}{ccc} 2 & 2 & 2 \\ 1 & 2 & 2 \\ -1 & 5 & -7 \\ 1 & -1 & 5 \end{array}\right],$$

$$\mathbb{L}(I,I) = \left[\begin{array}{cccc} -2 & -6 & -14 & 1 \\ 0 & -2 & -6 & 0 \\ 4 & 0 & 16 & -2 \\ 0 & 4 & 0 & 0 \end{array}\right], \ \mathbb{L}_s(I,I) = \left[\begin{array}{cccc} 1 & -1 & -5 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 5 & 1 & 1 \\ 1 & 1 & 5 & 1 \end{array}\right].$$

Since condition (18) is satisfied with $r = 4$ and the rank of the Loewner matrix is 3, we recover a minimal descriptor realization with incorporated $\mathbf{D}$ term:

$$\mathbf{W}(:,I)(\mathbb{L}_s(I,I) - s\mathbb{L}(I,I))^{-1}\mathbf{V}(I,:) = \mathbf{H}(s).$$

An alternative way to obtain the interpolant is to project the original matrices by randomly generated projectors:

$$\mathbf{Y}^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix}, \ \mathbf{X} = \begin{bmatrix} 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & -1 \end{bmatrix} \quad \Rightarrow$$

$$\mathbf{E}_\delta = \begin{bmatrix} -19 & 10 & 21 & 19 \\ -\frac{31}{2} & 3 & \frac{9}{2} & \frac{15}{2} \\ \frac{565}{8} & -\frac{103}{4} & -\frac{687}{8} & -\frac{641}{8} \\ \frac{45}{4} & -\frac{35}{2} & -\frac{95}{4} & -\frac{65}{4} \end{bmatrix}, \ \mathbf{A}_\delta = \begin{bmatrix} 12 & 6 & 1 & -4 \\ 14 & 18 & 16 & 5 \\ -\frac{63}{4} & \frac{41}{2} & \frac{153}{4} & \frac{127}{4} \\ -\frac{25}{2} & -5 & \frac{5}{2} & \frac{15}{2} \end{bmatrix},$$

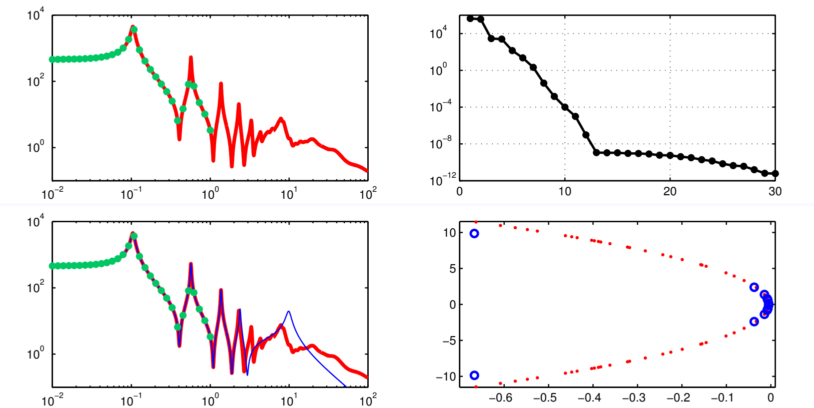
$$\mathbf{B}_\delta = \begin{bmatrix} 1 & -3 & 17 \\ 1 & 4 & 12 \\ -1 & 24 & -54 \\ 0 & 5 & -15 \end{bmatrix}, \mathbf{C}_\delta = \begin{bmatrix} \frac{65}{8} & \frac{45}{4} & \frac{37}{8} & -\frac{13}{8} \\ \frac{29}{4} & \frac{17}{2} & \frac{25}{4} & \frac{3}{4} \end{bmatrix}.$$

$\Rightarrow \mathbf{C}_\delta(\mathbf{A}_\delta - s\mathbf{E}_\delta)^{-1}\mathbf{B}_\delta = \mathbf{H}(s)$.

We thus obtain a different (equivalent) descriptor realization with incorporated **D** term. ■

Example: a discretized Euler-Bernoulli beam

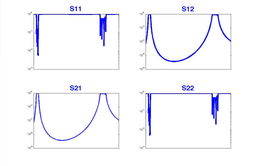
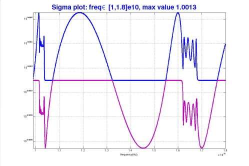
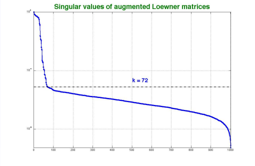
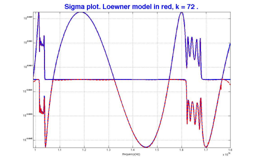
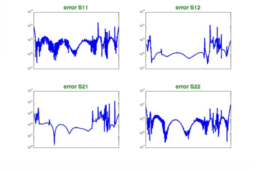
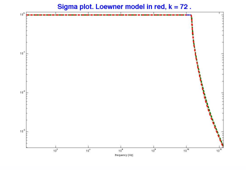
- System of order $n = 348$ (obtained after discretization) representing a clamped beam.
- $N = 60$ frequency response measurements, $s_k = j\omega_k$, with $\omega_k \in [-1, -0.01] \cup [0.01, 1]$.
- Construct 30×30 Loewner pencil and $\mathbf{Y}, \mathbf{X} \in \mathbb{R}^{30\times 12}$ from the SVD.
- Project to get reduced model of order $r = 12$.



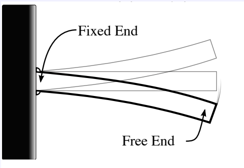
(1,1) Original and data	(1,2) Singular values of $\mathbb{L}$.
(2,1) Original & reduced FR	(2,2) Poles original & reduced

Reduced model from frequency response measurements

1001 S-parameter measurements between 10-18 GHz (CST).

Data frequency response $\|\mathbf{S}_{i,j}\|$, $i, j = 1, 2$.	Data two singular values.
	
Singular values of 1001 × 1001 Loewner matrix	Singular-value fit of model $k = 72$
	
$\mathbf{S}$-parameter-error: $\in [10^{-6}, 10^{-4}]$	Two singular values of model: $\omega \in [0, 10\text{THz}]$
	

An Euler-Bernoulli beam



$$\text{BC} \begin{cases} w(0,t)=0,\ \frac{\partial w}{\partial x}(0,t)=0,\ E\,I\,\frac{\partial^2 w(L,t)}{\partial x^2}+c_d I\,\frac{\partial^3 w(L,t)}{\partial x^2 \partial t}=0, \\ -E\,I\,\frac{\partial^3 w(L,t)}{\partial x^3}-c_d I\,\frac{\partial^4 w(L,t)}{\partial x^3 \partial t}=u(t),\ y(t)=\frac{\partial w(L,t)}{\partial t}, \end{cases}$$

$$\frac{\partial^2 w(x,t)}{\partial t^2}+\frac{\partial^2}{\partial x^2}\left[E\,I\,\frac{\partial^2 w(x,t)}{\partial x^2}+c_d I\,\frac{\partial^3 w(x,t)}{\partial x^2 \partial t}\right]=0,$$

where E, I, c_d are constants. The transfer function is:

$$\boxed{\mathbf{H}(s)=\frac{s\,\mathbf{N}(s)}{(E\,I+s\,c_d I)\,m^3(s)\,\mathbf{D}(s)}} \qquad \text{with} \quad \mathbf{m}(s)=\left[\frac{-s^2}{E\,I+c_d\,I\,s}\right]^{\frac{1}{4}},$$

$$\mathbf{N}(s)=\cosh(L\,\mathbf{m}(s))\sin(L\,\mathbf{m}(s))-\sinh(L\,\mathbf{m}(s))\cos(L\,\mathbf{m}(s)) \quad \text{and}$$

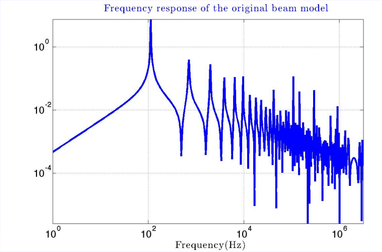
$$\mathbf{D}(s)=1+\cosh(L\,\mathbf{m}(s))\cos(L\,\mathbf{m}(s)).$$

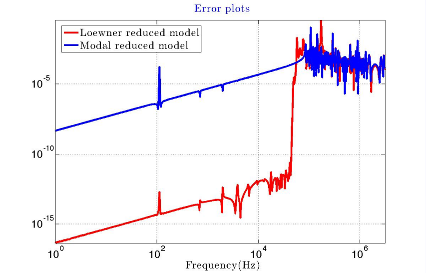
Parameter values: $E=69,\ GPa=6.9\cdot 10^{10}N/m^2$ - Young's modulus elasticity constant, $I=(1/12)\cdot 7\cdot 8.5^3\cdot 10^{-11}m^4$ - moment of inertia, $c_d=5\cdot 10^{-4}$ - damping constant, $L=0.7m$, $b=7cm$, $h=8.5mm$ - length, base, height of the rectangular cross section.

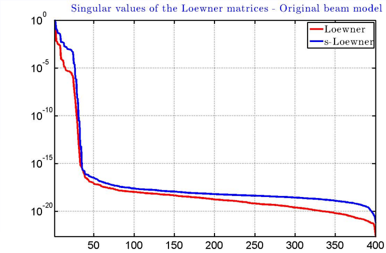
Plots

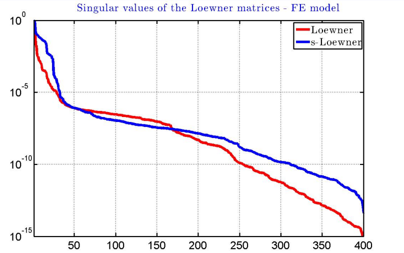
Reduction methods:

1. **Modal truncation.**
2. **FEM followed by Loewner.**
3. **Loewner based on the transfer function.**









Outline

1 Interpolatory model reduction

2 The Loewner framework
- Generalized inverses and rectangular systems
- Illustrative examples
- The Loewner algorithm
- Data-driven model reduction: bypassing PDE discretization

3 Time-domain modeling and model reduction

4 Summary and References

Chronobiology and rhythms

- Circadian (= circa + diem) or daily (24h) rhythms allow organisms to anticipate and prepare for environmental changes and best capitalize on environmental resources (e.g. light and food).
- The 24h circadian cycle has been well described. It is involved in many metabolic processes. Distortion of the 24 hour cycle has profound impacts on health.
- Our goal was to find other cycles (e.g. 12h) of gene transcription (which is a signal from the gene to a biological system).
- Recall that tides are mainly a 12h phenomenon.

Processing gene data

The data: finite records of $y_1, \cdots, y_N$, resulting form gene transcription or RER (Respiratory Exchange Ratio) measurements.

Basic model: sum of exponentials. Find $\alpha_i,\ \beta_i \in \mathbb{C},\ i = 1, 2, \cdots, k,$ such that

$$\mathbf{y}(t) = \mathbf{y}^*(t) + \mathbf{w}(t) \quad \text{where} \quad \mathbf{y}^*(t) = \sum_{i=1}^{k} \alpha_i\, e^{\beta_i t},$$

and $\mathbf{w}(t)$ is the noise. Requirement: $\mathbf{y}(i) \approx y_i,\ \ i = 1, 2, \cdots, N.$

Alternative formulation: descriptor representation, using an *internal variable* $\mathbf{x}(t) \in \mathbb{R}^k$:

$$\mathbf{E}\mathbf{x}(n+1) = \mathbf{A}\mathbf{x}(n), \quad \mathbf{y}(n) = \mathbf{C}\mathbf{x}(n) + \mathbf{w}(n), \quad \text{with initial condition} \quad \mathbf{x}(0) = \mathbf{x}_0,$$

where $\mathbf{E}$, $\mathbf{A} \in \mathbb{R}^{k\times k}$, $\mathbf{x}_0 \in \mathbb{R}^k$, $\mathbf{C} \in \mathbb{R}^{1\times k}$.

Processing of the data. Assume $N = 2k$; the data is used to form a **Hankel matrix**:

$$\mathcal{H} = \begin{bmatrix} y_1 & y_2 & y_3 & \cdots & y_{k-1} & y_k & y_{k+1} \\ y_2 & y_3 & y_4 & \cdots & y_k & y_{k+1} & y_{k+2} \\ y_3 & y_4 & y_5 & \cdots & y_{k+1} & y_{k+2} & y_{k+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ y_{k-1} & y_k & y_{k+1} & \cdots & y_{2k-3} & y_{2k-2} & y_{2k-1} \\ y_k & y_{k+1} & y_{k+2} & \cdots & y_{2k-2} & y_{2k-1} & y_{2k} \end{bmatrix} \in \mathbb{R}^{k\times(k+1)}.$$

Define the quadruple $(\mathbf{E}, \mathbf{A}, \mathbf{x}_0, \mathbf{C})$: $\mathbf{x}_0 = \mathcal{H}(1\!:\!k, 1) \in \mathbb{R}^k$, $\mathbf{C} = \mathcal{H}(1, 1\!:\!k) \in \mathbb{R}^{1\times k}$, and

$\mathbf{E} = \mathcal{H}(1\!:\!k, 1\!:\!k)$, $\mathbf{A} = \mathcal{H}(1\!:\!k, 2\!:\!(k+1)) \in \mathbb{R}^{k\times k}$, **(E, A) : Loewner pencil in time domain**

This quadruple constitutes the **raw (untreated)** model of the data.

This model is linear, time-invariant, discrete-time, **obtained with NO computation**:

$$\mathbf{E}\,\mathbf{x}(n+1) = \mathbf{A}\,\mathbf{x}(n), \quad \mathbf{y}(n) = \mathbf{C}\,\mathbf{x}(n), \quad \mathbf{x}(0) = \mathbf{x}_0.$$

Reduced models and fundamental oscillations: the **dominant** part of the raw system is determined using a **projection**: $\Pi = \mathbf{Y}(\mathbf{X}^T\mathbf{Y})^{-1}\mathbf{X}^T \in \mathbb{R}^{r\times r} \Rightarrow \mathbf{E}_r = \mathbf{X}^T\mathbf{E}\mathbf{Y}$, $\mathbf{A}_r = \mathbf{X}^T\mathbf{A}\mathbf{Y}$, $\mathbf{C}_r = \mathbf{C}\mathbf{Y}$, $\mathbf{x}_r = \mathbf{X}^T\mathbf{x}_0$. The associated reduced model of size r is:

$$\mathbf{E}_r\,\mathbf{x}_r(n+1) = \mathbf{A}_r\,\mathbf{x}_r(n), \quad \mathbf{y}_r(n) = \mathbf{C}_r\,\mathbf{x}_r(n), \quad \mathbf{x}_r(0) = \mathbf{x}_r$$

- **Orthogonality**. The fundamental oscillations are (almost) orthogonal $\mathbf{f}_i \perp \mathbf{f}_j$, $i \neq j$.
  - **Interpretation of orthogonality**. Orthogonality means that the fundamental oscillations are **independent** of each other.
  - Existing approaches: MUSIC, ESPRIT, Prony's method and statistical methods are compared with the new method described above, called **pencil method**. **NO** other method yields the **orthogonality relationships**.

Results: RER for restrictively fed mice (218 measurements every 40min)

Mouse #1		
A	*P*	*T*
0.0116	0.9961	7.4236
0.0256	0.9993	7.9961
0.0817	1.0001	23.9264
0.8843	1.0001	*dc*

Mouse #2		
A	*P*	*T*
0.0196	1.0008	7.9904
0.0072	0.9999	12.3797
0.0866	1.0001	23.8401
0.8913	0.9998	*dc*

Mouse #3		
A	*P*	*T*
0.0081	1.0023	6.0960
0.0170	1.0015	7.9728
0.0722	1.0010	23.8297
0.8941	0.9999	*dc*

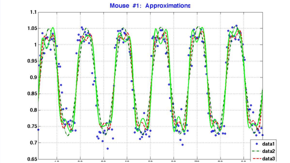
Mouse #4		
A	*P*	*T*
0.0151	1.0010	7.9423
0.0185	0.9995	12.2039
0.0904	0.9998	23.7697
0.9236	0.9997	*dc*

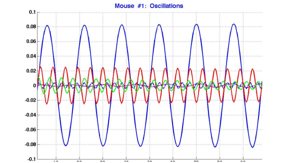
Mouse #5		
A	*P*	*T*
0.0220	1.0000	7.9621
0.0841	1.0004	23.8567
0.0080	0.9978	55.5220
0.8897	0.9998	*dc*

Mouse #6		
A	*P*	*T*
0.0149	1.0009	7.9535
0.0283	0.9930	12.4346
0.0891	1.0005	23.8193
0.9412	0.9997	*dc*

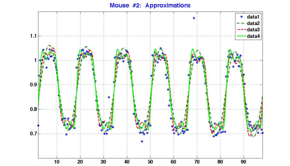
Approximation by 1, 2 and 3 oscillations | **First 4 oscillations**

Mouse #1



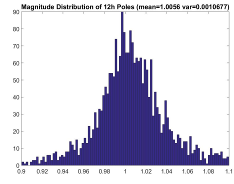


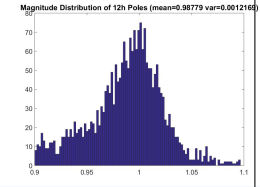
Mouse #2

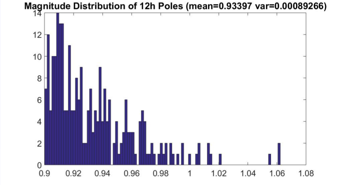


Results: comparison of methods applied to gene data

Pencil method	ESPRIT method	Prony's LS method
2354 genes	2350 genes	265 genes
$\mu = 1.005 - \sigma = 0.001$	$\mu = .988 - \sigma = 0.001$	$\mu = .934 - \sigma = .001$

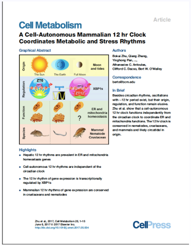






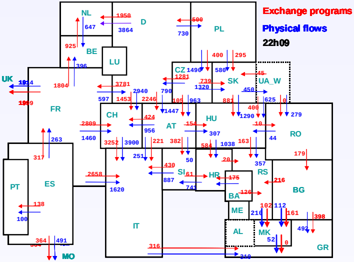
Conclusions

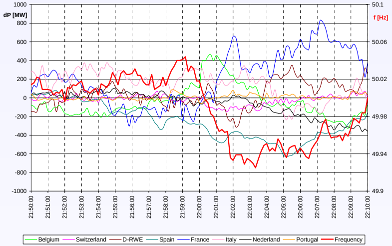
1. The prevalence of 12h oscillations is **1 in 8 genes**
2. The relation of the 12h with 24h rhythm: **Independent**
3. Reference: Bokai Zhu, Qiang Zhang, Yinghong Pan, Emily M. Mace, Brian York, Athanasios C. Antoulas, Clifford C. Dacso, and Bert W. O'Malley, *A cell-autonomous mammalian 12-hour clock coordinates metabolic and stress rhythms*, Cell Metabolism, June 2017.



The European blackout: 4 November 2006

Power exchange	Deviations of the power exchange and the frequency





Power exchange deviations in West area

λ (poles)	$\lvert\lambda\rvert$	σ (decay)	T (period)
$0.36297 \pm 0.79436i$	0.8734	-0.13541	5.501
$0.70901 \pm 0.64879i$	0.9610	-0.03973	8.478
$0.88856 \pm 0.44051i$	0.9917	-0.00827	13.652
$0.99516 \pm 0.17878i$	1.0111	0.01102	35.348
$-0.46003 \pm 0, 94443i$	1.0505	0.04927	3.104

Outline

1. Interpolatory model reduction
2. The Loewner framework
   - Generalized inverses and rectangular systems
   - Illustrative examples
   - The Loewner algorithm
   - Data-driven model reduction: bypassing PDE discretization
3. Time-domain modeling and model reduction
4. Summary and References

Summary: MOR in the Loewner framework

- Given is: time-domain, frequency domain measured or simulated data (DNS)

	Examples		Measurements
1.	clamped beam	$n = 346$	freq. domain
2.	semi-conductor device	no model	freq. domain
3.	Nonlinear Heat eqn	$n = \infty$	freq. domain
3.	Burgers equation	$n = \infty$	freq. domain
4.	Gene data	no model	time domain
5.	European blackout	no model	time domain

- Key tool: **Loewner pencil** (followed by a **projection**).

Types of systems
Linear (SISO and MIMO)
Linear parametrized
Linear switched systems
Bilinear
General quadratic bilinear

- Given data, we construct with **no computation**, a singular, high-order model in descriptor form ⇒ natural way to construct reduced models.

⇒ **SVD of the Loewner pencil provides trade-off between accuracy and complexity; resulting model complexity** $n =$ **(numerical) rank of Loewner matrix**.

⇒ can deal with **nonlinear systems**.

- Philosophy: **Collect data and extract desired information**

Some references

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