Tensors

Exercises

Exercise 1 (Matricization)

Let $\mathcal{X} \in \bigotimes_{j=1}^{3} \mathbb{R}^3$. The entries of $\mathcal{X}$ are given by x_{j_1,j_2,j_3} for $j_1, j_2, j_3 \in \{1,2,3\}$. Compute the following matricizations.

a) $\mathcal{X}^{\{1\}}$

b) $\mathcal{X}^{\{1,3\}}$

c) $\mathcal{X}^{\{2,3\}}$

d) $\mathcal{X}^{\{1,2,3\}}$

Exercise 2 (Equivalence Tucker and hierarchical Tucker format for $d \leq 2$)

The Tucker format and the hierarchical Tucker format are equivalent for orders $d = 1$ and $d = 2$. For $d > 2$ they are not. Discuss why by comparing the cases $d = 2$ and $d = 3$. For instance, write down the explicit tensors in the two formats for a tensor $v \in \mathbb{R}^n \otimes \mathbb{R}^n$ and $w \in \mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^n$.

Exercise 3 (r-term format)

Prove that the set of tensors in the r-term format

$$\mathcal{R}_r(\bigotimes_{j=1}^{d} V_j) := \{\sum_{k=1}^{r} \bigotimes_{j=1}^{d} v_k^j \mid v_k^j \in V_j \ \forall k \in \{1,\dots,r\}, \forall j \in \{1,\dots,d\}\}$$

is not closed for $r \geq 3$, $d \geq 3$ and V_j such that $\dim(V_j) \geq 2$ for all $j \in \{1,\dots,d\}$.
Hint: Consider pairs of vectors $v_j, w_j \in V_j$ that are linearly independent for $j \in \{1,2,3\}$. Now

$$X := v_1 \otimes v_2 \otimes w_3 + v_1 \otimes w_2 \otimes v_3 + w_1 \otimes v_2 \otimes v_3 \in \mathcal{R}_3$$

and find a series $s_n \in \mathcal{R}_2$ with $\lim_{n\to\infty} s_n = X$.

Exercise 4 (Comparison of tensor formats)

Is there any advantage representing an r-term tensor in the Tucker or the tensor train format for $d = 2$? On which circumstances does your answer depend? For instance, find a representation of the r-term tensor

$$v = \sum_{j=1}^{r} v_i \otimes w_i \text{ with } v_i, w_i \in \mathbb{R}^n$$

in the Tucker and the tensor train format and discuss your observations. For this, write v as a matrix first.

Exercise 5 (Hierarchical Tucker format)

In [Hackbusch '12] the kth basis element at a node α that is not a leaf is given recursively by

$$b_k^\alpha = \sum_{\substack{i\in\{1,\dots,r_{\alpha_1}\}\\ j\in\{1,\dots,r_{\alpha_2}\}}} c_{i,j}^{(\alpha,k)} (b_i^{\alpha_1} \otimes b_j^{\alpha_2}) \text{ where } C^{(\alpha,k)} \in \mathbb{R}^{r_{\alpha_1}\times r_{\alpha_2}} \tag{1}$$

where α_1 and α_2 are children of α. In [KressnerTobler '12] this relation is described by the equation

$$D_\alpha = (D_{\alpha_1} \otimes D_{\alpha_2}) E_\alpha \tag{2}$$

with bases D_α, D_{α_1}, D_{α_2} and a transfer matrix E_α. Try to understand both notations. How do D_α, D_{α_1}, D_{α_2} and E_α depend on b_k^α and $C^{(\alpha,k)}$?

Exercise 6 (htucker toolbox for Matlab)

Download the htucker Matlab toolbox. Add the unzipped toolbox folder to your Matlab path using `addpath`.

htucker - A MATLAB toolbox for tensors in hierarchical Tucker format
C. Tobler and D. Kressner, Switzerland, 2012
`http://anchp.epfl.ch/htucker` [accessed 2018/06/12]

Get familiar with

- constructing a tensor using `htensor()`
- truncating a full tensor to an htucker object using `truncate()`
- truncating an htucker object to an htucker object of lower rank.

For instance, set up three random vectors v_1, v_2, v_3 of size $n \in \mathbb{N}$ using the Matlab routine `rand()` and try to construct an htucker object X such that

`full(X)=kron(v3,kron(v2,v1)).`

Now set up another htucker object Y based on three other random vectors. Build the sum

`Z=X+Y.`

What can you say about the rank of X, Y and Z? Construct an htensor object in

$$\bigotimes_{j\in\{1,\dots,d\}} \mathbb{R}^n$$

that uses a binary tree and one that uses the TT partition tree.

Exercise 7 (TT toolbox for Matlab)

Download the tensor-train toolbox.

TT-Toolbox
V. Oseledets, S. Dolgov, V. Kazeev, D. Savostyanov,
O. Lebedeva, P. Zhlobich, T. Mach and L. Song, 2011
`https://github.com/oseledets/TT-Toolbox` [accessed 2018/06/12]

In order to use this toolbox one has to run

```
setup.m
```

in the directory of the TT-toolbox. The commands that are important here are

- `tt_tensor()` to truncate a full tensor to a TT object and
- `round()` to truncate a TT object to a TT object of lower rank.

In analogy to exercise 5 set up three vectors v_1, v_2, v_3 of size $n \in \mathbb{N}$ and construct a TT object such that

```
full(X)=kron(v3,kron(v2,v1)).
```

Truncate a random tensor of order $d = 2$ using `tt_tensor()`. Try to understand what the constructor does by taking a look at the source code `full.m` and the object property `tt.core`.

References

[1] D. Kressner and C. Tobler, Algorithm 941: `htucker`–a Matlab toolbox for tensors in hierarchical Tucker format, ACM Trans. Math. Software, **40(3)**, Art. No. 22, (2014).

[2] W. Hackbusch, Tensor spaces and numerical tensor calculus, (Springer, Heidelberg, 2012).