Lorenzo Baldi
Sebastian Debus

2. Day 20.06.2023

Exercise 1

Prove the following facts about the convex cone of Sums of Squares Σ^2.

0. If $\sigma \in \Sigma^2 \subset \mathbb{R}[\mathbf{x}] = \mathbb{R}[x_1, \dots, x_n]$, then $\sigma(\mathbf{x}) \geq 0 \ \forall \mathbf{x} \in \mathbb{R}^n$.
1. If $\sigma = h_1^2 + \cdots + h_r^2 \in \Sigma^2$ with $\max_{i \in \{1,\dots r\}} \deg(h_i) = d$, then $\deg(\sigma) = 2d$.
2. Σ^2 is a *pointed* convex cone, i.e. $\Sigma \cap -\Sigma = \{0\}$.
3. $\Sigma^2 - \Sigma^2 := \{ \sigma_1 - \sigma_2 \mid \sigma_1, \sigma_2 \in \Sigma^2 \} = \mathbb{R}[\mathbf{x}]$.

Exercise 2

Show that every univariate non-negative polynomial can be written as a sum of *two* squares.

Hint: use the Fundamental Theorem of Algebra.

Exercise 3

Show that the Motzkin polynomial $M(x, y) = 1 - 3x^2y^2 + x^2y^4 + x^4y^2$ is non-negative on $\mathbb{R}^2$ using the Arithmetic Mean-Geometric Mean inequality.

Exercise 4

Let $M(\mathbf{g})$ be the quadratic module generated by the tuple of polynomials $(g_1, \dots g_m)$. Prove that the following are equivalent.

(i) $\exists h \in M(\mathbf{g})\colon S(h) = \{ \mathbf{x} \mid h(\mathbf{x}) \geq 0 \}$ is compact.

(ii) $\exists r \in \mathbb{N}\colon r - \sum_{i=1}^n x_i^2 = r - \|\mathbf{x}\|_2^2 \in M(\mathbf{g})$.

(iii) $\forall f \in \mathbb{R}[\mathbf{x}], \exists r \in \mathbb{N}\colon r \pm f \in M(\mathbf{g})$.

These are the equvalent definitions of *Archimedean* quadratic module.

Hint: Show directly that (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). For (i) $\Rightarrow$ (iii) use Schmüdgen's Positivstellensatz.

Exercise 5 (Jabobi-Prestel Example)

Let $g_1 = x_1 - 1/2, g_2 = x_2 - 1/2, g_3 = 1 - x_1x_2$. Show that $S(\mathbf{g}) \subset \mathbb{R}^2$ is compact but $M(\mathbf{g})$ is not Archimedean (notice that $T(\mathbf{g})$ is Archimedean from Schmüdgen's Positvstellensatz).

Hints. Assume that $r - x_1 = \sigma_0 + \sigma_1 g_1 + \sigma_2 g_2 + \sigma_3 g_3 \in M(g_1, g_2, g_3)$ for some $r \in \mathbb{N}$. Consider the terms of homogeneous maximal degree: $\sigma_0^H, \sigma_1^H x_1, \sigma_2^H x_2, -\sigma_3^H x_1 x_2$. Among them, pick the ones of highest degree. Then prove the following.

- *If the terms with maximal degree are σ_0^H (or $-\sigma_3^H x_1x_2$), then $\sigma_0^H - \sigma_3^H x_1 x_2 = 0$ (since $r - x_1$ has odd degree). Prove that this implies $\sigma_0^H = \sigma_3^H = 0$, contradicting the hypothesis of maximal degree.*
- *Otherwise, in a similar way show that $\sigma_1^H x_1$ and $\sigma_2^H x_2$ cannot cancel (i.e. $\sigma_1^H x_1 \neq \sigma_2^H x_2$). Therefore $\sigma_1^H x_1 + \sigma_2^H x_2 = -x_1$. One can show that such a representation is not possible.*

We conclude that $r - x_1 \notin M(\mathbf{g})$, proving that $M(\mathbf{g})$ is not Archimedean.

Exercise 6

Given an ideal $I \subset \mathbb{R}[\mathbf{x}]$, show that $(I + \Sigma^2) \cap -(I + \Sigma^2)$ is an ideal of $\mathbb{R}[\mathbf{x}]$ and that:

$$\{\, f \in \mathbb{R}[\mathbf{x}] \mid \exists k \in \mathbb{N}, \sigma \in \Sigma^2 \text{ s.t. } f^{2k} + \sigma \in I \,\} = \sqrt{(I + \Sigma^2) \cap -(I + \Sigma^2)}$$

These are two equivalent definitions for the *real radical* of I, denoted $\sqrt[\mathbb{R}]{I}$.